In the name of God, the one, the Father.
Not Holy Spirit nor the Son.

Mathematical Induction

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Why is Mathematical Induction very important?

- A very powerful proof technique for discrete parts of mathematics.
- A powerful tool to design algorithms.
- One of fundamentals of discrete mathematics

What is Mathematical Induction?

- A specific but very important method of proving mathematical theorems.
- MI can be used to prove well ordering principle as a theorem or vice versa: Have well ordering principle as a principle and prove MI as a theorem.
- Well ordering principle says any non-empty subset of non negative numbers has got a smallest element. E.g. {5, 6, 7} has got 5.

What does MI say?

- MI says if we have got a subset of N such as A such that A has 1 and for every a in A, we have a+1 in A, we surely have A = N.
- In mathematical formal way:

$$A \subseteq \mathbb{N} \wedge 1 \in A \wedge (\forall a \in A \rightarrow a+1 \in A) \rightarrow A = N$$

How to use MI?

- We have a statement P which we want to prove it is valid for all non negative numbers.
- We first prove $P(1) \equiv T$
- Then we prove for every true P(a) we have P(a+1) true.
- Congrats! P(n) is true for every integer n starting from 1.

Example usage part 0

- We have a discrete function $f(n) = (n/2)(n+1)$
- We want to prove that f gives us the sum of all integers from 1 up to n.

Example usage part 1

- First we prove 1 = f(1)
- Since f(1) = (½) * (1+1) = (½) * 2 = 1 it is proven
- Now we prove considering f(a) gives the correct result, so does f(a+1).

Example usage part 2

$$\frac{n}{2}(n+1)=1+2+\ldots+n$$

$$1+2+\ldots+n+(n+1)=\frac{n}{2}(n+1)+(n+1)=\frac{n^2+n}{2}+n+1$$

$$\ldots=\frac{n^2+n}{2}+\frac{2n}{2}+\frac{2}{2}=\frac{n^2+2n+2}{2}=\frac{(n+1)(n+2)}{2}=f(n+1)$$

Other types of MI

- Odd Even MI
- Backward MI
- MI with a different starting point
- Spiral MI
- Double MI
- Triple MI

Odd even MI

- Prove that P(1) is true and also P(2)
- Prove that if P(a) is true so is P(a+2)
- P(n) is true for any positive n

A variant of Backward MI

- Prove that P(1) is true
- Prove that for every a, P(2^a) is true
- Prove that for if P(k) is true so is P(k-1)

MI with a different starting point

- This version of MI does not prove the statement for all positive numbers. It rather proves that P(n) is true for all positives from the starting point.

- You must prove for a as the starting point, P(a) is true instead of proving P(1) is true.

Spiral MI

- Using Spiral MI you can prove two statements, Q and P at the same time.
- Prove that P(1) is true
- Prove that if P(a) is true for some a in N so is Q(a)
- Prove that if Q(k) is true so is P(k+1)
- Congrats! P(n) and Q(n) are true for all positive numbers.

Double and Triple MI

- Double MI is Mathematical Induction using two variables m and n for such a statement. E.g. you've got P(m, n) and you want to prove that it is true for all m and n in N.
- Triple MI is a version of MI for statements with 3 arguments.

Double MI

- Prove P(1, 1) is true
- Prove if P(1, a) is true so is P(1, a+1)
- Prove if P(b, 1) is true so is P(b+1, 1)
- Congrats! P(m, n) has been proven true for all positive m and n.

Triple MI

- Just like double MI but with 3 variables.
- In summary, when we want to prove P(m, n, r) we must prove that P(1, 1, 1) is true.
- Then we must prove if P(a, 1, 1) is true so is P(a+1, 1, 1)
- And if P(1, b, 1) is true so is P(1, b+1, 1)
- And if P(1, 1, c) is true so is P(1, 1, c+1)
- The same for P(a, b, 1) and P(a+1, b, 1), P(a, b, 1) and P(a, b+1, 1), and so on.
- Finally you'll prove that P(m, n, r) is valid for any positive m, n and r.

A way to imagine MI

- For single variable MI imagine a building. If you can build the first block and for any block if you can build a block on it, you can surely build the whole building.
- For double MI, you can imagine a small 2D space with discrete values.
- For triple MI, you can imagine a 3D space with discrete values.

How to use MI to design algorithms?

- MI can be used as a method to design many algorithms, too.
- An example is the sorting problem: Given an ordered sequence of numbers, how can we sort them from the lowest to highest?
- There are many many algorithms for this problem.
- Using MI we can design several algorithms for the sorting problem.
- Given n integers which we want to sort them, we assume there is a way to to sort n-1 items and we want to know where to put the nth integer.

Further work and study

- Use the game Minetest(https://minetest.net) to build 1D, 2D and 3D buildings.
- https://qcweb.qc.edu.hk/math/Resource/AL/Different%20kinds%20of%20Mathematical%20Induction.pdf for different types of MI plus exercises for each type with solutions.
- Using induction to design algorithms(14P): http://akira.ruc.dk/~keld/teaching/CSS_e10/Manber88.pdf
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Thanks for your time and effort